Homodyne detection, Wigner function

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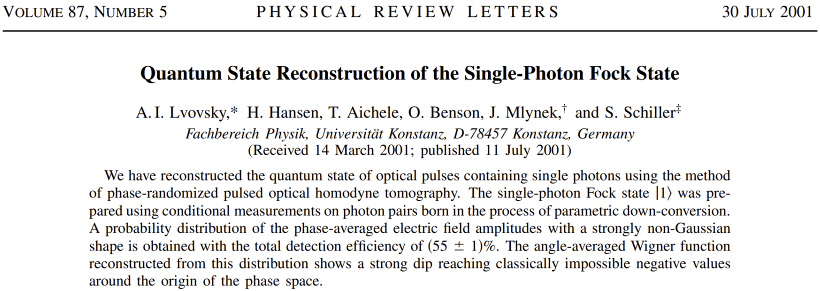
Introduction and Paper objective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Quantum State Reconstruction of the Single-Photon Fock State

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We have reconstructed the quantum state of optical pulses containing single photons using the method of phase-randomized pulsed optical homodyne tomography. The single-photon Fock state $|1\rangle$ was prepared using conditional measurements on photon pairs born in the process of parametric down-conversion. A probability distribution of the phase-averaged electric field amplitudes with a strongly non-Gaussian shape is obtained with the total detection efficiency of $(55 \pm 1)\%$. The angle-averaged Wigner function reconstructed from this distribution shows a strong dip reaching classically impossible negative values around the origin of the phase space.

Figure: Quantum reconstruction of the single-Photon Fock state, July 2001

Optical Homodyne Tomography

Is a Method used long before this paper to determine the quantum state of an optical wave, the main challenge here is the preparation of the $|1\rangle$ state.

Wigner function

Definition

$$W(p,q) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx}\left\langle q - \tfrac{x}{2}\middle|\,\hat{\rho}\,\middle|q + \tfrac{x}{2}\right\rangle dx$$

Marginal distributions

$$\langle p|\hat{\rho}|p\rangle = |\psi(p)|^2 = \int_{-\infty}^{\infty} W(p,q)\, dq \qquad \langle q|\hat{\rho}|q\rangle = |\psi(q)|^2 = \int_{-\infty}^{\infty} W(p,q)\, dp$$

This is the main definition formulated by Wigner in 1932. As we can see from the equations. we are dealing with the position and momentum operators q and p respectively. For an electromagnetic field, we cannot use position and momentum operators. Thus, we define quadratures.

Wigner function

Optical Quadratures

$$\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger)$$

$$\hat{X}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger)$$

$$[X_1, X_2] = \frac{i}{2}$$

or more generally:

$$X_\varphi = \hat{a}e^{i\varphi} + \hat{a}^\dagger e^{-i\varphi}$$

and the conjugate quadrature would be $X_{\varphi + \frac{\pi}{2}}$

Wigner function

The phase-space quasi-probability density

$$\Pr(q_\theta, \theta) = \int_{-\infty}^{\infty} W\Big(q_\theta \cos\theta - p_\theta \sin\theta,\ q_\theta \sin\theta + p_\theta \cos\theta\Big)\, dp_\theta.$$

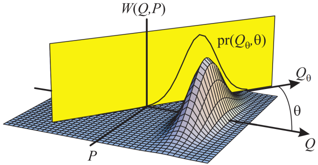

Figure: Phase space quasi-probability density

The experimentally measured probability density is the integral projection of the Wigner function onto a vertical plane defined by the phase of the local oscillator.

Homodyne detection

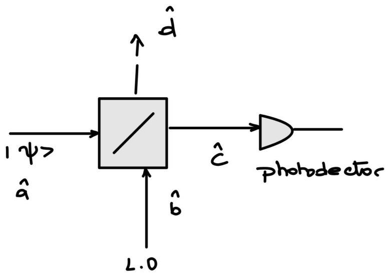

Figure: Homodyne detection

Photocurrent

$$I(t) \propto < \hat{c}\hat{c}^{\dagger} > = Constant - i\sqrt{\eta(1-\eta)}\beta < X_{\phi} >$$

Homodyne tomography

- Homodyne detection gives us access to the average phase quadrature

Homodyne measures :

$$\left\langle X_\phi \middle| \hat{X} \middle| X_\phi \right\rangle = P_\phi(X_\phi)$$

- By repeating this measurement for different angles, We can reconstruct the Wigner function.
- for a rotationally symmetric Wigner function (which is the case for Fock states) we can construct the Wigner function using the Abel transform

Abel transform

$$W(R) = \frac{2}{\pi} \int_R^\infty \frac{d\,p_{\text{av}}(X)}{dX} \frac{dX}{\sqrt{X^2 - R^2}}.$$

Experimental Setup and resulting states

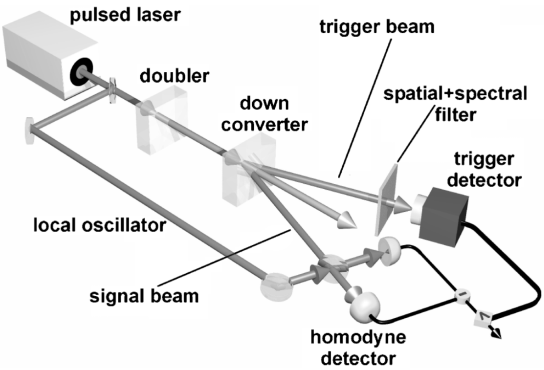

Figure: Experimental setup

Resulting states

- A non linear crystal pumped by laser produces photon pairs into different optical modes

Generated bi-photon State

$$|\Psi\rangle = N\left(|0,0\rangle + \int d\vec{k}_s\, d\vec{k}_t\, \Phi(\vec{k}_s,\vec{k}_t)\, |1_{\vec{k}_s}, 1_{\vec{k}_t}\rangle\right).$$

- A single photon- fock state is prepared from $|\Psi\rangle$ by projecting this state on a photon count event in the trigger beam path.

Resulting signal beam

$$\hat{\rho}_s = T_{r_t}[|\Psi_{out}\rangle\langle\Psi_{out}|\,\hat{\rho}_t]$$

state selection operator

$$\hat{\rho}_t = \int d\vec{k}_t T(d\vec{k}_t)\left|1_{\vec{k}_t}\right\rangle\left\langle 1_{\vec{k}_t}\right|$$

Measured States

- Once the state $\hat{\rho}_s$ is prepared,it is subject to a Homodyne detection
- the measured state after the Homodyne detection is labeled $\hat{\rho}_{meas}$
- Two cases should be distinguished here :

Ideally

$$\hat{\rho}_{meas} = |1\rangle \langle 1|$$

Real experiment

$$\hat{\rho}_{meas} = \eta |1\rangle \langle 1| + (1-\eta) |0\rangle \langle 0|$$

- η is the measurement efficiency taking into account all effects such as optical losses in the signal arm, inefficient photodiodes, dark counts, non-ideal matching between the signal and the LO optical modes

Reconstruction

In this experiment, the phase ϕ varies randomly, so that we measured only a single phase-randomized marginal distribution. This does not impact the result since for Fock states the Wigner function is rotationally symmetric. $p_{\text{av}}(X) = \langle p(X_\phi)\rangle_\phi$

Computing the Wigner function

$$W(R) = \frac{2}{\pi}\int_R^\infty \frac{d\,p_{\text{av}}(X)}{dX}\frac{dX}{\sqrt{X^2 - R^2}}.$$

We can then reconstruct the density matrix :

Density operator reconstruction

$$\rho_{nn} = \pi \int_{-\infty}^{\infty} p_{\text{av}}(X)\, f_{nn}(X)\, dX,$$

$f_{nn}(X)$ are the amplitude pattern functions

Results (1)

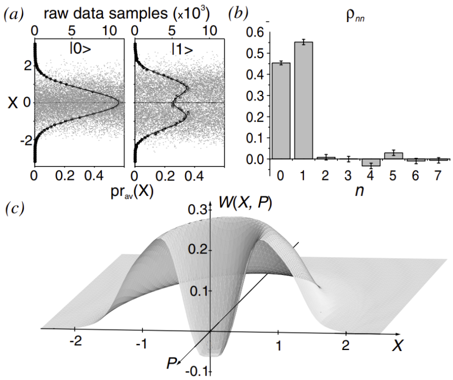

Figure: results obtained

As we can see the result exhibits a negative value in the origin a symbol of non classicality of the fock states.

Results (2)

- The best fit efficiency value $\eta = 0.55 > 0.5$
- With the Abel transform we get $W(0,0) = -0.062$
- $\rho_{11} = 0.553 \pm 0.013$, In agreement with η
- if we calculate the corresponding Wigner function using the density matrix we get : $W(0,0) = -0.067 \pm 0.016$

Proposed Questions and Answers

Question 1

What's the expression of Wigner function they used ?

They use the Abel transform to compute the Wigner function with the measured phase randomized marginal distribution :

$$W(R) = \frac{2}{\pi} \int_R^\infty \frac{d\, p_{\text{av}}(X)}{dX} \frac{dX}{\sqrt{X^2 - R^2}}.$$

Question 2

What measurement data is taken and how is it used to reconstruct the Wigner function

They perform balanced Homodyne detection of field quadratures. For many local-oscillator (LO) phases θ, they collect samples of $X_\theta = X\cos\theta + P\sin\theta$ and build the marginal distributions $p_r(X_\theta)$. These marginals are Radon projections of the Wigner function $W(X,P)$ and can be inverted to reconstruct W.

Proposed Questions and Answers

Question 3

How does measurement efficiency impact the result ?

All imperfections (losses, detector quantum efficiency, mode mismatch, dark counts) appear as an admixture of vacuum:$\hat{\rho}_{\text{meas}} = \eta \left|1\right\rangle\left\langle 1\right| + (1-\eta) \left|0\right\rangle\left\langle 0\right|$ Reduced efficiency η smooths the marginal distributions and lifts $W(0,0)$ toward zero; observing negativity requires $\eta > 0.5$. Their fit gives $\eta \simeq 0.55$, and they reconstruct $W(0,0) \approx -0.062$.

Question 4

How does signal-LO mode-matching influence efficiency in homodyne detection ?

Mode mismatch reduces the effective efficiency by the visibility squared of the interference between LO and the modeled signal mode. They achieved a visibility $\nu \approx 0.83$ contributing to a factor $\nu^2 \approx 0.69$ Additional imperfect modeling of the signal(heralded) mode adds a factor of 0.95.

Proposed Questions and Answers

Question 5

Why use a single laser for both LO and (indirectly) the signal path ?

Using the same laser ensures that the local oscillator and the signal photons are phase-coherent and have identical spectral and temporal characteristics. This greatly reduces mode mismatch in Homodyne detection. If two independent lasers were used, even tiny frequency or phase drifts would destroy the interference needed to measure field quadratures.

Question 6

Why do they use a doubler and down-converter for single photon sources

The 790 nm laser light is first frequency-doubled to 395 nm to serve as the pump for spontaneous parametric down-conversion (SPDC) in the nonlinear crystal. Each pump photon at 395 nm splits into two lower-energy photons (signal and trigger) at 790 nm. This way, the generated single photons have the same wavelength as the local oscillator, ensuring frequency matching for homodyne detection.

Proposed Questions and Answers

Question 7

How does the spatial-temporal pulse shape of single photon match LO ?

Directly matching a heralded single photon is hard, so they back-pump the crystal along the trigger path with alignment pulses to generate difference-frequency generation (DFG) that closely models the signal's mode. They then overlap this DFG field with the LO and maximize interference visibility, which sets the spatial–temporal match used for the actual single-photon measurements.

Proposed Questions and Answers

Question 8

How is the density matrix reconstructed? What are the values of the off-diagonal terms?

From the phase-randomized marginal distribution $p_{\text{av}}(X)$, the diagonal elements of the density matrix are computed in the Fock basis using pattern functions:

$$\rho_{nn} = \sqrt{\pi} \int_{-\infty}^{\infty} p_{\text{av}}(X) f_{nn}(X)\, dX.$$

Because the local oscillator phase is randomized, the state becomes rotationally symmetric in phase space. Averaging over all phases eliminates the phase factors $e^{-i(m-n)\theta}$ for $m \neq n$, so all off-diagonal terms ρ_{mn} vanish. Hence, only the diagonal elements ρ_{nn} can be reconstructed. Experimentally, $\rho_{11} = 0.553 \pm 0.013$, consistent with $\eta \approx 0.55$.

Proposed Questions and Answers

Question 9

The paper mentions that within 14 hour of experiment 200 000 measurements corresponded to a vacuum state in the signal mode (no click of the PD), and only 14 000 corresponded to a single photon state in the signal mode (PD clicked). Can the fraction of useful measurement outcomes be increased by increasing the intensity of the pump laser impinging on the up conversion crystal thus increasing the amplitude of the coherent state in the single mode?

If the pump power is increased, the pair production rate also increases, resulting in more trigger events and thus more heralded single-photon detections per unit time. However, a higher pump power also raises the probability of generating multiple photon pairs within the same pulse. In that case, the signal state is no longer a pure single-photon state $|1\rangle$. The signal beam is not a coherent state; ideally it is a Fock state $|1\rangle$. The local oscillator (LO) is the coherent state, obtained by splitting a small portion of the original laser pulses before the frequency doubling stage. the fraction of useful measurement will therefore not increase and the accuracy of the measurement will be affected by the generation of the multi-photon pairs.